EMPIRICAL STUDY OF A COMPETENCE STRUCTURE MODEL REGARDING CONVERSIONS OF REPRESENTATIONS – THE CASE OF FRACTIONS

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Given the key role of conversions of representations for mathematical understanding, it is highly relevant to investigate in detail competencies regarding conversions of representations. In particular, a corresponding competence model should not only be developed theoretically, but also examined empirically. However, such empirical studies are rather scarce, especially concerning content domains other than functions. Consequently, this study focuses on the design and empirical validation of a competence structure model regarding conversions of representations in the domain of fractions using multidimensional item response modelling. The results suggest that the data support the theoretically developed structure of the model and moreover, they indicate a hierarchical relationship which may give rise to a competence level model.

INTRODUCTION

The ability of dealing flexibly with distinct representations of a mathematical concept and changing between them has been shown to be an important factor for successful mathematical thinking and problem solving (e.g. Lesh, Post & Behr, 1987; Deliyianni et al., 2008). Research into students' competencies regarding the idea of using multiple representations is thus highly relevant. Hence, our project "La viDa-M" (c.f. Dreher, Winkel & Kuntze, 2012) aims at investigating aspects of students' competence regarding conversions of representations domain-specifically by focusing on the content domain of fractions. Moreover, La viDa-M examines possible impact factors on such competencies including specific professional knowledge and views of their teachers. Central to the first project phase is the development of a competence model for learners and its empirical evaluation, on which we will report in this paper. Taking into account different research projects and findings concerning students' competencies in dealing with multiple representations, a competence model regarding conversions of representations and a corresponding domain-specific test instrument were designed. In order to validate the developed model empirically the data of 675 students in 29 sixth-grade classes were analyzed using multidimensional item response modelling. The theoretical background, methods and results reported in the following refer to this first phase of our project. In the last section, additionally to the discussion of these results an outlook is given on first findings regarding interrelations between students' specific competencies and teachers' corresponding views.

2014. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 2, pp. 425-432. Vancouver, Canada: PME.

THEORETICAL BACKGROUND

The significance of using multiple representations for learning mathematics is emphasized in many national standards (e.g. KMK, 2003; NCTM, 2000). This has good reasons: Doing mathematics relies on using representations, since mathematical objects are not accessible without them (Duval 2006). In fact, a single representation standing for a mathematical object is usually not enough, since mostly a representation can merely emphasize some properties of the corresponding object, so multiple representations have to be integrated in order to develop appropriate conceptual understanding (Ainsworth, 2006; Duval, 2006). Consequently, making connections and conversions between different representations is central to the understanding of mathematical concepts (e.g. Lesh, Post & Behr, 1998; Deliyianni et al., 2008, Renkl et al., 2013). For the purposes of this study we chose to focus on conversions of representations in the content-domain of fractions, since it is particularly well-known that different representations of fractions may highlight different core aspects of the concept and that hence changing between them is important (e.g. Ball, 1993).

This key role of conversions of representations for conceptual understanding leads to the research aim of describing learners' competence regarding conversions of representations. Two requirement scenarios can be distinguished: Firstly, a conversion of representations may be given, which has to be *examined*, i.e. one has to check whether two representations match, if they represent the same mathematical object. Secondly, a conversion of representations may have to be *performed*, i.e. one has to construct a matching second representation in a different representation register on the base of a given representation. Similar distinctions have been made by several researchers investigating students' competencies in dealing with multiple representations, who focused however mostly on the content domain of functions (c.f. e.g. Hitt, 1998, Bossé, Adu-Gyamfi & Cheetha, 2011, Nitsch et al., accepted). Bossé et al. (2011) differentiate for instance between "interpretative activity" and "constructive activity" and Nitsch et al. (accepted) use the distinction of "identification" and "construction" referring to them as "elements of cognitive action". However, in the cited studies it becomes not entirely clear whether the notions "interpretative activity" resp. "identification" refer to single representations or to conversions of representations. Yet, it makes a difference whether aspects of one given representation have to be identified/interpreted or if a conversion of representations has to be examined in the sense of identifying/interpreting aspects of both given representations and deciding if they match. Since we focus on learners' competencies regarding conversions of representations, we do not adopt these notions, but use instead the terms *examining a conversion* and *performing a conversion*. As metacognitive activities like justifying, in the sense of reflecting, explaining and giving reasons play an important role for conceptual understanding using multiple representations (c.f. Renkl et al., 2013), learners should also be able to justify why a given or a self-performed conversion of representations is correct or not. Regarding the content domain of functions, Nitsch et al. (accepted) have implemented the actions "description" and "explanation" in their competence structure model, which could however not be separated empirically, but formed a common dimension instead. With respect to the domain of fractions Deliyianni et al. (2008) differentiated between "recognition tasks" and "conversion tasks" within the construct of "flexibility in multiple representations" and they have also taken into account so-called "justification tasks", but those were operationalized as being part of another competence construct, namely "problem solving". However, seeing the ability to justify conversions of representations as an important facet of competence regarding dealing with multiple representations, it appears to be appropriate to include it into the structural modelling of such competence. Hence, our theoretical competence structure model regarding conversions of representations encompasses the following facets: *examining, performing* and *justifying*. In particular, tasks regarding conversions of representations may require *examining* or *performing* these conversions and optionally they may in addition ask for *justifying* the given or self-performed conversions. Since it may be argued that these three abilities differ in their cognitive demands, this suggests a 3-dimensional competence model (3D) regarding conversions of representations which is shown in Figure 1. According to this model *examining*, *performing* and *justifying* of conversions of representations form one dimension each in the sense of being empirically separable (but not necessarily independent) constructs representing different facets of such competence.

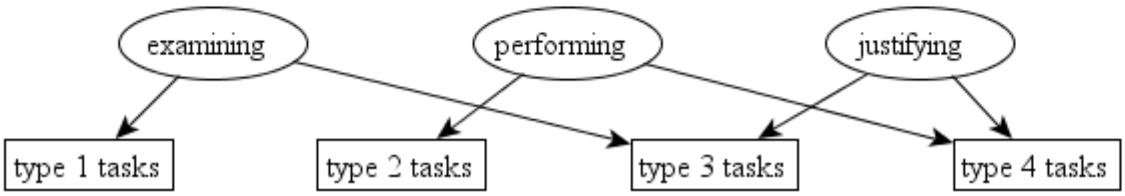


Figure 1: 3D competence structure model regarding conversions of representations

For the purpose of empirical validation of the structure of this model, multidimensional item response theory (MIRT) is used, which is particularly suitable for psychometric modelling of competence taking into account different potentially relevant abilities (Hartig & Höhler, 2008). In this approach possible alternative models are compared to the anticipated model (c.f. Figure 1) with respect to how well the empirical data from our study focusing on the domain of fractions fit them. One of these alternative psychometric models which should be taken into account is the 2-dimensional model (2D), where *examining* and *performing* are not separated, but form a common dimension. This dimension is hence relevant for all tasks regarding conversions of representations and *justifying* represents a separate (optionally relevant) dimension, as it requires metacognition which has to be verbalized. Moreover, a 1-dimensional psychometric model (1D) which assumes that a single dimension represents all three abilities regarding conversions of representations should be tested.

Besides the structure of the competence regarding conversions of representations in the sense of underlying dimensions, the level of difficulty of the abilities encompassed are highly relevant for designing specific learning opportunities and for the diagnosis of learning processes. From a theoretical point of view one may suppose that *performing* a conversion of representations is generally more difficult than *examining* a given conversion of representation, since in the first case a new representation has to be created (c.f. Nitsch et al., accepted). Corresponding assumptions can be found for instance in the context of the theoretical competence level model by Hitt (1998). Since empirical evidence for such a hierarchy is however still lacking, it is a question worth investigating, whether *performing* is generally speaking more difficult than *examining* with respect to conversions of representations.

RESEARCH INTEREST

Examining the model shown in Figure 1 in comparison with other potential models in the content domain of fractions can help to describe the structure of competence regarding conversions of representations. In line with the need for research outlined above, the evaluations presented in this paper are guided by the following research questions:

- Is it possible to validate our theory-based competence structure model regarding conversions of representations in the domain of fractions empirically using multidimensional item response theory?
- Do the empirical data support the theoretical assumption that *performing* conversions of representations constitutes a higher level of difficulty than *examining* conversions of representations?

DESIGN, SAMPLE AND METHODS

For answering these research questions, a test instrument corresponding to our theoretical competence structure model was designed specifically for the domain of fractions. In line with the structure shown in Figure 1, this competence test includes four types of tasks, for each of which Table 1 shows a sample item. The first type is about *examining* given conversions regarding their correctness, i.e. one has to decide if given representations match in the sense of representing the same mathematical object. The second type of tasks demands *performing* conversions of representations. For solving the third resp. fourth type of tasks, it is not enough to examine resp. perform conversions of representations, but they also have to be justified. From each type, three tasks were included in the test instrument, so that it consisted of 12 items in total. Tasks of different types were arranged in alternating order. The paper-pencil test was completed by 675 students in 29 sixth-grade classes at academic track secondary schools in Germany. Within a lesson (45 min.) they were given enough time to solve all the tasks under the supervision of a member of the project team. The answers to each task were scored dichotomously as being correct or incorrect according to criteria established beforehand. Prior to fitting any item response models, one of the type 2 tasks which had been revised after piloting had to be excluded, as a misconception could lead to a correct answer of the item. The modelling of the competence structure was conducted with CONQUEST software (Wu et al, 2007) using multidimensional item response theory (Rasch analysis).

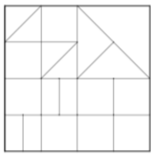
Examining a conversion (type 1 tasks)	*Examining* a conversion and *justifying* (type 3 tasks)
The second photo was taken shortly after the first one. 1.  2.  Do the following calculations match what has happened between the two shots? A. $\frac{6}{8}+\frac{1}{8}$ ☐ yes ☐ no B. $\frac{7}{8}-\frac{1}{8}$ ☐ yes ☐ no	Class 6c has lost a soccer game 3-4 against class 6d . Lars considers whether the class 6c has scored $\frac{3}{4}$ or $\frac{3}{7}$ of the total goals. Fatima wants to help Lars: "Only a fraction less than $\frac{1}{2}$ is possible." Is Fatima right? ☐ yes ☐ no Why or why not?
Performing a conversion (type 2 tasks)	*Performing* a conversion and *justifying* (type 4 tasks)
For solving word problems you have to find calculations to given situations. Here you are asked to do it the other way round. Write down a word problem which exactly matches the calculation $\frac{2}{2}+\frac{1}{2}$.	 Take two crayons and color parts of the square so that the calculation $\frac{2}{16}+\frac{2}{16}$ is shown and the entire square is the whole. Explain in detail why the calculation can be seen in your representation.

Table 1: Sample items for each of the four types of tasks

RESULTS

Focusing on the first research question, we started by fitting the three possible models (1D, 2D and 3D) to the data. Table 2 shows the resulting deviances as a measure of discrepancy and the number of parameters estimated as a measure for the complexity of the model. Since models using more parameters always deviate less (or at least equally) from the real data, both these characteristics of the models have to be taken into account for deciding which one fits best. As the 1D model is a sub-model of the 2D model, which requires two parameters less, the difference between the deviances of the two models follows an approximate chi-square distribution with two degrees of freedom (c.f. Wu et al., 2007). Given the estimated difference of 20.6 in the deviance, we conclude that the extra parameters of the 2D model highly significantly improve the fit ($p<.001$). In the same way we can compare the 2D model with the 3D model, as the 2D model is a sub-model of the 3D model with three fewer parameters estimated. Considering the chi-square distribution with three degrees of freedom shows that the reduction in deviance of 12.36 indicates that the 3D model may fit the data significantly better than the 2D model ($p<.05$).

Model	1D	2D	3D
Deviance	7533.83	7513.23	7500.87
# Parameters	12	14	17

Table 2: Comparison of the fits of the three alternative models

The evaluation of how well the items in the developed competence test fit these models, can be done based on the weighted mean square (MNSQ) fit statistics (c.f. Wu et al., 2007). As this statistic takes the value 1 for a perfectly fitting item, we have checked for each item whether its weighted MNSQ statistic regarding the respective model is significantly different from 1. This analysis shows for the 1D model that not all the MNSQ fit statistics lie inside the ninety-five percent confidence interval for the expected value and thus we have rejected the null hypothesis that the data conforms the model. For the 2D model as well as for the 3D model however, the weighted MNSQ statistic of none of the items is significantly different from 1 ($0.90 \leq MSNQ \leq 1.07$, resp. $0.95 \leq MSNQ \leq 1.06$), which indicates that the test items fit both of these models very well.

Addressing our second research question, we focus next on comparing the difficulties of the tasks which demand *examining* conversions with those demanding *performing* conversions of representations. The difficulties estimated from the data which are displayed in Figure 2 indicate that in both cases (with or without requirement of justifying) *performing* was more difficult than *examining* with respect to conversions of representations in the domain of fractions. The same pattern could also be found by considering simply the percentage of students who have solved the respective items.

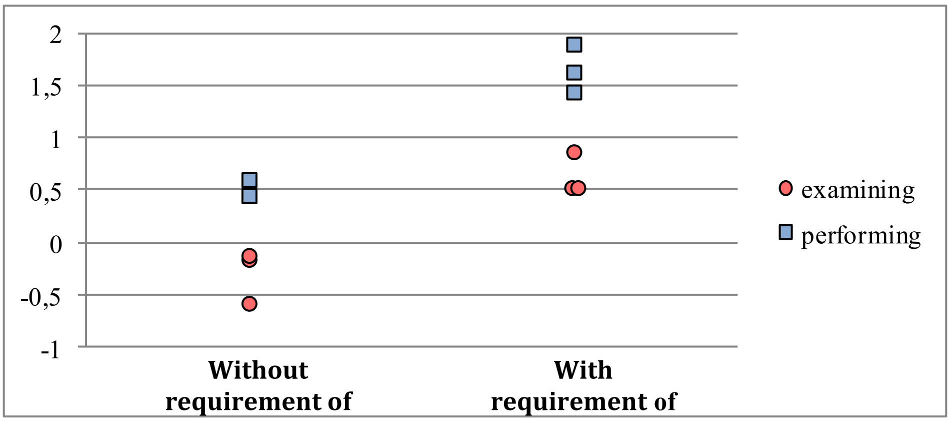


Figure 2: Empirical difficulties of the tasks of the four different types

DISCUSSION AND OUTLOOK

The results of this study may contribute to a better understanding of the construct of competence regarding conversions of representations – with respect to its structure as well as with respect to the differentiation of possible competence levels.

Before these results are discussed in more detail we would however like to recall the limitations of this study which suggest interpreting the evidence with care: Although the sample of this study is reasonably large, it is not representative for German students in sixth grade. Furthermore, even though a spectrum of different items was developed according to the theoretical competence structure model, only a relatively small number of items could be implemented in the test instrument for reasons of feasibility.

Bearing this in mind, the findings however allow answering the research questions and indicate several aspects of theoretical and practical relevance.

Concerning the first research question, the result that the 3D model fits the data better compared to the alternative models backs up the structure of our theoretical competence model regarding conversions of representations. Moreover, seen in connection with similar findings by Nitsch et al. (accepted) with respect to the domain of functions, this indicates that the framework may even be valid across content domains. The finding that the items also fit the 2D model very well suggests that the 2D model, where *examining* and *performing* conversions of representations form a common dimension, may also be used for pragmatic and simplicity reasons. It has the advantage that a joint competence score for both of these abilities may be considered.

Regarding our second research question the results have provided some empirical evidence for a hierarchical relationship of the abilities *examining* and *performing* which was previously merely theoretically postulated. This finding may be an important step towards a model of competence levels regarding conversions of representations and hence it should be replicated by studies using a bigger pool of items and also focusing on additional content domains. From a practical point of view, implications of the findings of this study concern in particular the design of specific learning opportunities, the analysis of the demands of tasks and the diagnosis of learning processes with respect to conversions of representations (in the domain of fractions).

First evaluations focusing both on students' competencies regarding conversions of fractions as well as on their teachers' views on how to use multiple representations for teaching fractions suggest interesting interrelations. For instance, the teachers' view that pictorial representations of fractions should merely be used for the introduction of the concept was significantly negatively related to the mean joint competence score (examining and performing conversions) of his or her students ($r=-.55$, $p<.01$). Despite such significant correlations, multi-level analysis showed that the differences between classes are not significant. This could be due to the fact that individual differences within the classes are much higher than the differences between the classes. However, further analyses have to be conducted in order to explore possible explanations for this interesting phenomenon.

Acknowledgements

The study presented in this paper is embedded in the project La viDa-M which is funded by a research grant from Ludwigsburg University of Education.

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